Donaldson. Introduction to Differential Geometry on toric varieties. 2013/6 SCGP (2 lectures).

§ General context.

X Kähler metrics, in a fixed cohomology class.

(Analog. pos. def. Herm. metrics.
Convex Cone $\subset \mathbb{R}^{k^2}$
$\simeq$ Symmetric space $GL(k,\mathbb{C})/U(k)$.)

$\omega_\varphi = \omega_0 + \partial\bar\partial\varphi > 0$ complex str. fixed

(X,ω) Symplectic (all forms are 'equiv'.)

Fix ω

$p \in X$, $\mathcal{J}_p = \{$compatible cx. str. on $T_pX\}$

compat. alm. cpx. str. $\longleftrightarrow$ section of $\mathcal{J}$

$\mathcal{G} \triangleq \{$Symplectomorphisms$\} \curvearrowright \mathcal{J}$

($\sim$ analog $U(k)$ above).

Space of Kähler metrics $= \underbrace{\mathcal{G}^{\mathbb{C}}}_{\text{does not exist!}} / \mathcal{G}$

$\rightsquigarrow$ geodesic in $\mathcal{G}^{\mathbb{C}}/\mathcal{G}$ ($\leftrightarrow$ homog. M.A. eqt.) Mabuchi.

In toric case, all these are much more visible.

§ Standard theory of toric manifolds

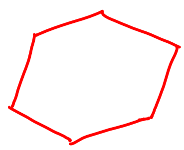
X compact Kähler

$T = (S^1)^n$ acts effectively of holom. isometries

Basic correspondence: Such data $\longleftrightarrow P \subset \mathbb{R}^n \supset \mathbb{Z}^n$
(P convex polytope)

P: $\lambda_r(x) \geqslant c_r$. Delzant.
(λ_r linear w/ $\mathbb{Z}$-coeff.)

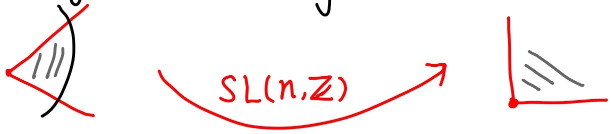
- vertices integral



(i.e. P is convex hull of finite # of lattice points).

Near each vertex P is defined by $\lambda_1 \geq c_1, \dots, \lambda_n \geq c_n$, say.

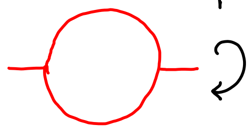
$\lambda_1, \dots, \lambda_n$ integral basis for $(\mathbb{Z}^n)^*$



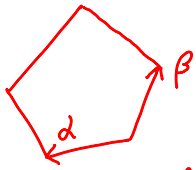
1) Complex manifolds $(T^n)^{\mathbb{C}} = (\mathbb{C}^\times)^n$

must act holomorphically, orbit open dense.

eg. $n = 1$ $\quad X = S^2$



$= \mathbb{CP}^1 = \mathbb{C}^\times \amalg \{0\} \amalg \{\infty\}$

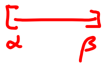


$g_{\alpha\beta} \in SL(n,\mathbb{Z})$ acts on $(\mathbb{C}^\times)^n$

$\log z_a' = \sum (g_{\alpha\beta})_{ab} \log z_b$, i.e. $z_a' = \prod z_b^{(g_{\alpha\beta})_{ab}}$

Vertex $\longrightarrow$ chart $\simeq \mathbb{C}^n$

glue over $(\mathbb{C}^\times)^n \subset \mathbb{C}^n$, using these maps

(eg. $\mathbb{CP}^1$ is $z' = \frac{1}{z}$ because  $g_{\alpha\beta} = (-1)$)

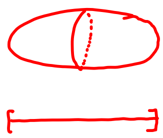
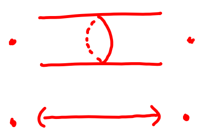
2) Symplectic

moment map $\mu : \overset{T^n \curvearrowright}{X^{2n}} \longrightarrow \mathbb{R}^n$

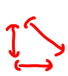
$P := \mu(X)$ is convex polytope in $\mathbb{R}^n$ (G-S, Atiyah).

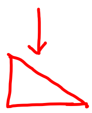
fibers are T^n-orbits.

$\mathrm{Int}\, P \times T^n \subset X$ (i.e. free orbits over P°)

 as 

$T^2 \subseteq SU(3) \overset{\text{std.}}{\curvearrowright} \mathbb{CP}^2$

$\mathbb{CP}^2 =$  $\times T^2 \amalg$  $\times T^1 \amalg$ $\vdots$.



Fixed pts: $[1,0,0], [0,1,0], [0,0,1]$,

$z_i = 0 \rightsquigarrow$ coord. $\mathbb{CP}^1$.

$P^\circ \times T^n$ w/ $\omega = \sum dx^i \wedge d\theta_i$, need to extend it.

3) Algebraic. L positive line bundle.

$\rightsquigarrow \bigoplus_{k \geq 0} H^0(X, L^k)$ graded ring

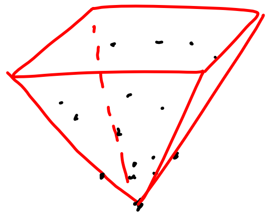
T^n acts $\rightsquigarrow$ decompose it.

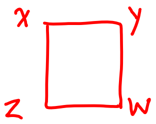
$P \subset \mathbb{R}^n$, $(kP \cap \mathbb{Z}^n)$

$\mathbb{Z}^{n+1} \supset A = \{(k,r) : r \in kP \cap \mathbb{Z}^n\}$ semi-group.

Conclusion: Each weight in $H^0(X, L^k)$ multi. 1

$\sim kP \cap \mathbb{Z}^n$





$\mathbb{CP}^1 \times \mathbb{CP}^1$ generator x, y, z, w, relation $xw = yz$

i.e. quadric in $\mathbb{CP}^3$

§ Differential Geometry

Complex coord. $\exp(t_a + i\theta_a)$ on $(\mathbb{C}^\times)^n \subset X$

Kähler potential $\varphi(\underline{t})$ $\quad \underline{t} \in \mathbb{R}^n$

$$i \partial\bar\partial \varphi \sim \frac{\partial^2 \varphi}{\partial t_a \partial t_b} =: \varphi^{ab}$$

$$\omega = \varphi^{ab} \, dt_a \wedge d\theta_b \qquad (\varphi \text{ convex fu. on } \mathbb{R}^n)$$

Symplectic view $\quad \underset{x^a}{P^0} \times \underset{\theta_a}{T^n} \quad \omega = \sum dx^a \wedge d\theta_a$

cpx. str. $\sim$ $d\theta_a + Z_{ab} \, dx^b$ as $(1,0)$-forms

$Z = (Z_{ab})$ cpx. matrix., $Z = Z^T$, $\operatorname{Im} Z > 0$

(Siegel upper half space)

Integrability. $\dfrac{\partial Z_{ab}}{\partial x^c} = \dfrac{\partial Z_{ac}}{\partial x^b}$

$\Rightarrow Z_{ab} = i \dfrac{\partial t_a}{\partial x^b}$ w/ $t_a = \dfrac{\partial F}{\partial x^a}$, $\exists$ cx. fu. $F(\underline{x})$.

$H(x) \rightsquigarrow$ symplectomorphism, commutes w/ T^n.

$$\rightsquigarrow \quad Z \longmapsto Z + \frac{\partial^2 H}{\partial x^a \partial x^b}$$

$$\Rightarrow \quad Z = i \frac{\partial^2 u}{\partial x^a \partial x^b} \quad \exists \ \underline{\text{real}} \text{ valued fu. } u(\underline{x}).$$

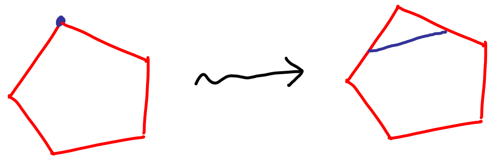
Summary.

cx.	sympl.
φ convex on $\mathbb{R}^n$	u convex on P°
$\omega = \sum \varphi^{ab} dt_a \, d\theta_b$	$\omega = \sum dx^a d\theta_a$ std
J std	$g = u_{ab} dx^a dx^b + u^{ab} d\theta_a d\theta_b$

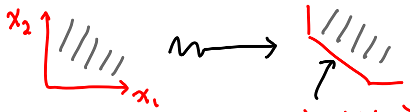
Legendre

§ Basic Toric Geometry (say $n=2$)

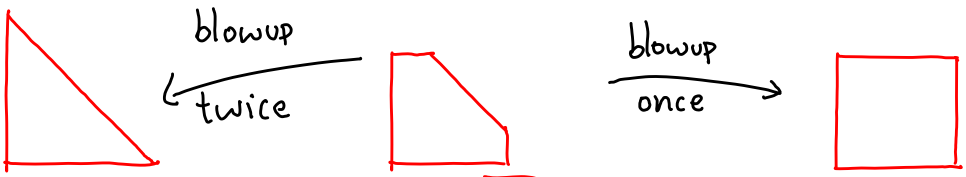
Blowing up a point



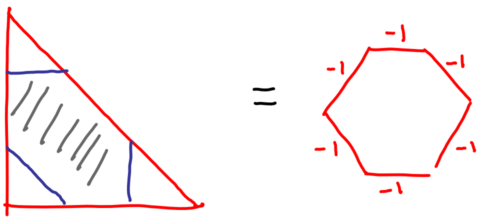
locally



$x_1 + x_2 \geq \varepsilon \sim \int_E \omega$ ex



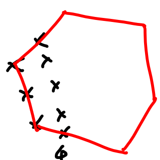
$\mathbb{CP}^2$ $\qquad$ $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ $\qquad$ $\mathbb{CP}^1 \times \mathbb{CP}^1$



$\widetilde{\mathbb{CP}^2}^{(3)}$

This admits KE metric.

When is X Fano ?



sect^s vanish to 1^st order.

$\mathbb{Z}$-coeff $\to$ $\lambda_r(0) = 0$ + $\lambda_r(x) \leq 1$.

$\sim \quad L = K_X^{-1} \qquad v_1 \wedge v_2$ as section

§ Differential Geometry $X \supset (\mathbb{C}^\times)^n \cong P^\circ \times T^n$

$\exp(t_a + i\theta_a)$

$\phi(t)$ convex as Kähler potential

$$\omega = \sum dx^i \wedge d\theta_i \quad \text{on} \quad P^\circ \times T^n$$

$u(x)$ convex. $u_{ab} = \frac{\partial^2 u}{\partial x^a \partial x^b}$, $(u^{ab}) = (u_{ab})^{-1}$

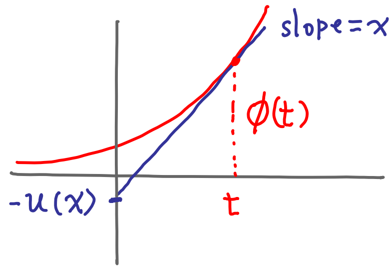
Legendre transform

$$\rightsquigarrow \quad g = u_{ab}\, dx^a dx^b + u^{ab}\, d\theta_a d\theta_b.$$

§ Legendre transform

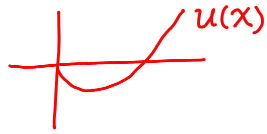
$\phi(t)$ on $\mathbb{R}^n$ & $u(x)$ on $(\mathbb{R}^n)^*$

$$u(x) + \phi(t) = tx \quad \text{w/} \quad x^a = \frac{\partial \phi}{\partial t^a}$$



Eg. $\phi(t) = e^t \Rightarrow \phi' = e^t = x \Rightarrow xt = u(x) + x$

$\Rightarrow u(x) = x \log x - x$ w/ $x \geq 0$



Eg. $\phi(t) = \log(e^t + e^{-t})$ $\quad t = \frac{1}{2} \log \frac{1+x}{1-x}$

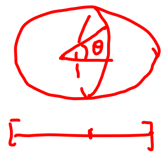
$u(t) = \frac{1}{2}\left((1+x)\log(1+x) + (1-x)\log(1-x)\right) + \log 2$

$0 < x < 1$ $\quad$ ($\rightsquigarrow$ std metric on S^2)

$$u'' = \frac{1}{1-x^2}$$

$$g = \frac{1}{1-x^2} dx^2 + (1-x^2)\, d\theta^2$$

$$= d\phi^2 + \sin^2\phi\, d\theta^2 \qquad x = \cos\phi$$



- $X \hookrightarrow \mathbb{P}(H^0(L^k)^*)$ $\rightsquigarrow$ restrict ω_{FS}

  base elt. $\leftrightarrow$ $r \in kP \cap \mathbb{Z}^n$

  $a_r > 0 \longrightarrow$ Kähler metric

$$\rightsquigarrow \phi(t) = \log\left(\sum_r a_r e^{rt}\right)$$

(Eg. $\mathbb{P}^1$, $O(1)$ $\rightsquigarrow$ $-1[\text{———}]1$ $\rightsquigarrow$ $\log(e^t + e^{-t})$)

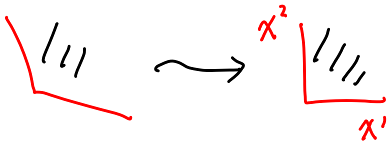
- $u^{ab}{}_{,kl} \sim$ Riemannian curv.

  Scalar curv. $S = u^{ab}{}_{,ab}$ (Abreu)

  $L \triangleq \log\det(u_{ij})$

  $$S = -(u^{ij} L_{,j})_{,i} = \Delta L$$

- Guillemin boundary condition:

  $u = x^1 \log x^1 + x^2 \log x^2 + \text{smooth}$

  defing $\lambda_r \rightsquigarrow$ measure on ∂P, $d\sigma$

- Const. Scalar curv. $S = \text{const.}$ $(= A)$ $\leftarrow$ fixed

  $$\delta \int_P \log\det(u_{ij}) = \int_P u^{ij} f_{,ij} \qquad f = \delta u$$

  $$= \int_P u^{ij}{}_{,ij} f + \int_{\partial P} \underbrace{u^{ij}{}_{,j}}_{\substack{(n-1)\text{form} \\ d\sigma \text{ measure}}} f \qquad (\because \text{ Bdy condition}).$$

  Say $f = 1 \Rightarrow \int_P S = \text{Mea}(\partial P, d\sigma)$ $\leftarrow$ fixed.

  Say $f = \text{linear} \Rightarrow$ moment of S is determined.

  $\rightsquigarrow$ nec. condit^{n} of $S \equiv \text{const.}$

- Mabuchi functional:

  $$\mathcal{M}(u) := -\int_P \underbrace{\log\det(u_{ij})}_{\Downarrow \text{ convexity}} + \underbrace{\int_{\partial P} u \, d\sigma - \int_P Au}_{\text{linear } \mathcal{L}_A(u)}$$

  Crit. pt. of $\mathcal{M} \longleftrightarrow R \equiv A$.

  $\exists!$ affine linear A st. $\mathcal{L}_A(f) = 0$, $\forall$ affine linear f

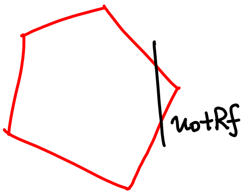
  linear part of $A \longleftrightarrow$ Futaki inv.

  Seek extremal metrics.

• (want $d\sigma$ evenly spread out on ∂P)

Condition S: $\mathcal{L}_A(f) \geq 0$ $\forall$ convex. f

> 0 $\forall$ affine linear f



(necessary condition)

Conjecture: Condit^{n} S $\Leftrightarrow$ $\exists$ extremal metric

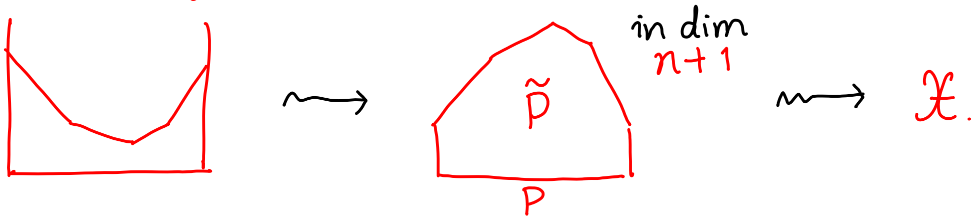
• $n = 2$ ✓

• Connection with algebraic geometry.

Stability of X $\sim$ $\mathcal{X} \xrightarrow{\pi} \Delta$ $\quad \pi^{-1}(t) \simeq X$, $t \neq 0$

$\pi^{-1}(0) \simeq X_0 \neq X$

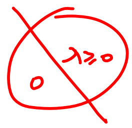
Suppose f $\mathbb{Q}$-PL fu.



Problems. 1) Extremal metrics $n > 2$?

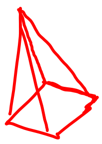
2) Rationality $\longleftrightarrow$ alg. geom ?

λ affine linear $\quad \lambda^+ := \max(\lambda, 0)$



$\Rightarrow$ $)_x \rightsquigarrow)$

3) Fano, but singular



$\sum_{i=1}^{4} z_i^2 = 0$

(asym. of metrics ?)